M337/Specimen exam 3

Module Examination

Complex analysis

There are **two parts** to this examination.

In **Part 1** you should **submit answers to all 6 questions**. Each question is worth 10% of the total mark.

In **Part 2** you should **submit answers to 2 out of the 3 questions**. Each question is worth 20% of the total mark.

Do not submit more than two answers for Part 2. If you submit answers to all three Part 2 questions, then only Questions 7 and 8 will be marked.

Include all your working, as some marks are awarded for this.

Write your answers in **pen**, though you may draw diagrams in pencil.

Start your answer to each question on a new page, clearly indicating the number of the question.

Crossed out work will not be marked.

Follow the instructions in the online timed examination for how to submit your work. Further information about completing and submitting your examination work is in the *Instructions and guidance for your remote examination* document on the module website.

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8.1

Part 1

You should **submit answers to all questions** from Part 1.

Each question is worth 10%.

Question 1

Let $w = -2 + 2i$.

(a) Determine each of the following complex numbers in *Cartesian* form, simplifying your answers as far as possible.

(i) $1/w$ [2]

(ii) $\operatorname{Log} w$ [2]

(iii) $\operatorname{Log}(w^3)$ [3]

(b) Find all the cube roots of w in *polar* form. [3]

Question 2

Locate the singularities of each of the following functions, and classify each singularity as a removable singularity, a pole (stating its order) or an essential singularity.

(a) $f(z) = \dfrac{\sinh z}{z}$ [3]

(b) $f(z) = \dfrac{\sin z}{(z-\pi)^3}$ [4]

(c) $f(z) = e^{1/(z-i)}$ [3]

Question 3

Show that the functions

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n \quad (|z| < 3)$$

and

$$g(z) = -\sum_{n=1}^{\infty} \left(\frac{3}{z}\right)^n \quad (|z| > 3)$$

are indirect analytic continuations of each other. [10]

Question 4

(a) Let f be the function

$$f(z) = 2z^3 + 3z^2 - 12z.$$

For each of the following points α, find an integer n such that f is n-to-one near α.

(i) $\alpha = 0$ [*2*]

(ii) $\alpha = 1$ [*3*]

(b) Prove that the series

$$\sum_{n=1}^{\infty} \frac{e^z}{n^2}$$

is uniformly convergent on the set $\{z : 0 \leq \operatorname{Re} z \leq 1\}$. [*5*]

Question 5

(a) Let q be the velocity function

$$q(z) = \frac{(\overline{z} + i)^2}{\overline{z}}$$

for an ideal flow with flow region $\mathbb{C} - \{0\}$. Use the Circulation and Flux Contour Integral to classify 0 as a source, sink or vortex of the flow. [*5*]

(b) Use the Joukowski function to find a one-to-one conformal mapping from the open punctured disc $\{z : 0 < |z| < 1\}$ onto $\mathbb{C} - [-2, 2]$. [*5*]

Question 6

(a) Show that the point $\alpha = e^{i\pi/11}$ is a periodic point of the function

$$f(z) = z^3.$$

Find the multiplier of the corresponding cycle, and determine whether the cycle is attracting, repelling or indifferent. [*6*]

(b) Find a real number c in the Mandelbrot set such that ic does not belong to the Mandelbrot set. [*4*]

Part 2

You should **submit answers to two questions** from Part 2. If you submit answers to all three Part 2 questions, then only Questions 7 and 8 will be marked.

Each question is worth 20%.

Question 7

(a) Let A and B be sets. Determine whether each of the following implications is true or false, either by proving the implication or by providing a counterexample.

- If A and B are compact, then $A \cap B$ is compact.
- If A and B are regions, then $A \cap B$ is a region.
- If A and B are compact, then $A \cup B$ is compact.
- If A and B are regions, then $A \cup B$ is a region. *[10]*

(b) Let f be the function

$$f(z) = z(1 + \overline{z}).$$

(i) Use the Cauchy–Riemann Theorem and its converse to show that f is differentiable at 0, but not analytic there. *[8]*

(ii) Evaluate $f'(0)$. *[2]*

Question 8

(a) Determine the radius of convergence of each of the following power series.

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{e^{in}}{n!}(z+i)^n$ *[3]*

(ii) $\displaystyle\sum_{n=0}^{\infty} (3^n + \cos n) z^n$ *[3]*

(b) (i) Find the Taylor series about 0 for the function

$$g(z) = \cos(z \exp z),$$

up to the term in z^4.

Explain why the series represents g on $\mathbb{C}$. *[5]*

(ii) Hence evaluate the integral

$$\int_C z^3 g(1/z)\, dz,$$

where C is the circle $\{z : |z| = 2\}$. *[4]*

(c) Suppose that f is an entire function that is bounded on the strip

$$\{z : 0 \le \operatorname{Re} z \le 1\},$$

and satisfies $f(z - n) = f(z)$, for all $n \in \mathbb{Z}$ and $z \in \mathbb{C}$.

Prove that f is a constant function. *[5]*

Question 9

Let f be the function

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{9z^2 + 1}.$$

(a) Find the Laurent series about 0 for f, giving the first two non-zero terms. [4]

(b) Find the residues of f at each of the points 0, $\frac{1}{3}i$ and $-\frac{1}{3}i$. [5]

(c) Hence determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{9n^2 + 1}.$$ [8]

(d) Use your solution to part (c) to prove that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{9n^2 + 1} = \frac{\pi}{3 \sinh \pi/3}.$$ [3]

[END OF QUESTION PAPER]